Homework 5
Continuum Mechanics

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April 25, 2021

Problem 1. In this problem, we are going to compare the relative importance between stretching and bending a beam. To do this, consider a long beam oriented along the positive z-direction that is clamped on its left-most point. Neglect gravity.

1. If we apply a stretching force F_z on the right-most point of the beam, show that the displacement at that point can be written as

$$u_z = \frac{F_z}{A}\frac{L}{Y},$$

with L the length of the beam, A the cross-sectional area and Y the Young's modulus.

2. If we instead apply a downward force (along the Y-direction) F_y on the right-most point of the beam, show that the displacement is

$$u_y(z = L) = \frac{L^3 F_y}{3YI} \approx \frac{L^3 F_y}{3YA^2},$$

where we have taken $I \approx A^2$ in the last step.

This implies:

$$\left|\frac{u_z}{u_y}\right| \approx \frac{A}{L^2}\left|\frac{F_z}{F_y}\right|.$$

Hence, for long beams ($A \ll L^2$) and comparable stretching and bending forces ($\left|\frac{F_z}{F_y}\right| \approx 1$), we see that $|u_z| \ll |u_y|$. This imples that the longitudinal displacement is always negligible compared to the transverse (due to bending) displacement.

Solution for a):
We have that $\sigma_{zz} = \frac{F_z}{A} = Yu_{zz}$ by Hooke's law, and all the other components of the stress vector are zero. Therefore, $u_{zz} = \frac{F_z}{AY}$.

Since $u_{zz} = \frac{du_z}{dz}$, we have

$$u_z = \int_0^R u_{zz} dz = \frac{F_z L}{AY}.$$

Problem 2. Consider a cylindrical beam of length L and cross-sectional radius R oriented along the positive $\hat{e}_z$ direction and made of an isotropic and homogeneous elastic material with Lame coefficients λ and μ. We fix the left-most end of the beam, which is located at $z = 0$, such that the diplacement vector in the corresponding cross-section is zero. The other end, located at $z = L$, is subjected to a force (per unit area) $\bar{\bar{\sigma}} \cdot \hat{e}_Z|_{r=R} = \sigma_L \hat{e}_\phi$, where $\bar{\bar{\sigma}}$ is the stress tensor, σ_L is a constant force per unit area supplied at $r = R$ to the circular cross-section located at $z = L$, and we are using the cylindrical system.

The resultant deformation results in a displacement vector that increases linearly with z, and that, based on the symmetry of the problem, we can write as

$$\vec{u} = (0, u_\phi(r, z), 0) = (0, R(r)Z(z), 0) = R(r)Z(z)\hat{e}_\phi,$$

where we have separated the (r, z)-dependence of u_ϕ into its r- and z-dependences. You can safely ignore body forces, such as the gravitational force.

1. Obtain the displacement field in the solid.
2. Obtain the strain tensor.
3. Obtain the stress tensor.
4. What is the total work done by the applied stress?

Solution for a):
Since $\vec{u}$ increases linearly with z, we have that $Z(z) = az + b$.

In order to find the displacement field, we will use Navier-Cauchy's equation combined with the general expression we have for the displacement field:

$$\underbrace{\vec{f}}_{=0} + \mu\nabla^2\vec{u} + (\lambda + \mu)\nabla(\nabla \cdot \vec{u}) = 0 \iff$$

$$\iff \mu(0, \partial_{rr}u_\phi + \partial_{zz}u_\phi + \frac{1}{r}\partial_r u_\phi - \frac{1}{r^2}u_\phi, 0) + (\lambda + \mu) \cdot 0 = 0 \iff$$

$$\iff \partial_{rr}u_\phi + \partial_{zz}u_\phi + \frac{1}{r}\partial_r u_\phi - \frac{1}{r^2}u_\phi = 0 \iff$$

$$\iff R''(r) + \frac{1}{r}R'(r) - \frac{1}{r^2}R(r) = 0 \iff$$

$$r^2R''(r) + rR'(r) - R(r) = 0$$

If we take as an ansatz $R(r) = C \cdot r^k$, we can clearly see that two independent solutions for the ODE which form the basis of the solution space are:

$$\left\{r, \frac{1}{r}\right\},$$

and so the general solution for the radial component is:

$$R(r) = cr + d\frac{1}{r}.$$

If we impose that when u_ϕ must be bounded near 0, we get that $D = 0$, and so in reality

$$R(r) = cr.$$

We also have that at $z = 0$, $u_\phi = 0 \quad \forall r$ since the beam is fixed, so:

$$0 = u_\phi(0) = R(r)(0 \ \cdot A + B) \implies B = 0.$$

Thus:

$$u_\phi = Acrz = Erz,$$

where $E := Ac$.

Solution for b):

$$\nabla\vec{u} = \begin{pmatrix} 0 & Ez & 0 \\ -Ez & 0 & 0 \\ 0 & Er & 0 \end{pmatrix} \implies \bar{\bar{u}} = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Er \\ 0 & Er & 0 \end{pmatrix}.$$

Solution for c):
We can get the stress tensor from the strain tensor via Hooke's law:

$$\bar{\bar{\sigma}} = 2\mu\bar{\bar{u}} + \lambda(\mathrm{Tr}\,\bar{\bar{u}})\bar{\bar{I}},$$

but since $\nabla \cdot \vec{u} = 0$, our expression simplifies to:

$$\bar{\bar{\sigma}} = 2\mu\bar{\bar{u}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & E\mu r \\ 0 & E\mu r & 0 \end{pmatrix}.$$

We are now in good position to impose the boundary condition of the stress tensor at the end of the beam ($z = L$):

$$\sigma_L \hat{e}_\phi = [\bar{\bar{\sigma}} \cdot \hat{e}_z]_{r=R} = \left[\begin{pmatrix} 0 \\ \mu E r \\ 0 \end{pmatrix}\right]_{r=R} = \mu E R \hat{e}_\phi \implies E = \frac{\sigma_L}{\mu R}.$$

Therefore:

$$\vec{u} = \left(0, \frac{\sigma_L}{\mu R} rz, 0\right),$$

$$\bar{\bar{u}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma_L}{\mu R} r \\ 0 & \frac{\sigma_L}{\mu R} r & 0 \end{pmatrix},$$

$$\bar{\bar{\sigma}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_L \frac{r}{R} \\ 0 & \sigma_L \frac{r}{R} & 0 \end{pmatrix}.$$

Solution for d):
We can calculate the elastic energy as:

$$u = \frac{1}{2}\bar{\bar{\sigma}} : \bar{\bar{u}} = \frac{\sigma_L^2}{2\mu}\frac{r^2}{R^2}.$$

Given that $u = \frac{W}{V}$, we can integrate the elastic energy over the entire volume of the beam to find the work done by the applied stress:

$$W = \int_V u\, dv = \frac{\sigma_L^2}{2\mu R^2}\int r^3 d\phi dr dz = \frac{\sigma_L^2}{2\mu R^2}\frac{2\phi L}{4R^2} = \frac{\sigma_L^2 \pi R^2 L}{4\mu}.$$

Problem 3. Consider a transverse plane wave, $\vec{u} = \vec{a}\exp\Big[i(\vec{k}\cdot\vec{r} - \omega t)\Big]$, propagating through an elastic solid. The solid is homogeneous, isotropic and has Lame coefficients λ and μ.

1. Show that the propagation of these waves does not involve volume changes.
2. Assume now that the wave propagates along the $\hat{x}$ direction. Calculate the stress tensor and justify why we call these waves, *shear waves.*
3. Now consider a longitudinal wave. Show that $\boldsymbol{\nabla} \times \vec{u} = 0$. Hint: You may consider using the Levi-Civita symbol.

Assume now that the wave propagates along the $\hat{x}$ direction. Confirm there are no off-diagonal terms in Cauchy's strain tensor. Hence, these waves propagate without shear distorsions; only normal stresses are involved. That's why we often refer to them as *pressure or compressional waves.*

Solution for a):

$$\nabla\vec{u} = \mathrm{Tr}\,\bar{\bar{\vec{u}}} = i\exp\Big[i(\vec{k}\cdot\vec{r} - \omega t)\Big]\vec{a}\cdot\vec{k},$$

as we calculate in the following subsection. But since $\vec{a}$ and $\vec{k}$ are orthogonal, we have that $\nabla\vec{u} = 0$, and therefore the volume of the elastic solid remains invariant.

Solution for b):

$$\bar{\bar{u}} = \frac{1}{2}[\nabla\vec{u}+(\nabla\vec{u})^t] = i\exp\Big[i(\vec{k}\cdot\vec{r} - \omega t)\Big]\frac{1}{2}\begin{pmatrix} 2a_xk_x & a_yk_x + a_xk_y & a_zk_x + a_xk_z \\ a_xk_y + a_yk_x & 2a_yk_y & a_zk_y + a_yk_z \\ a_xk_z + a_zk_x & a_yk_z + a_zk_y & 2a_zk_z \end{pmatrix} =$$

$$= i\exp\Big[i(\vec{k}\cdot\vec{r} - \omega t)\Big]\frac{1}{2}\left[\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}\begin{pmatrix} a_x & a_y & a_z \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}\begin{pmatrix} k_x & k_y & k_z \end{pmatrix}\right].$$

From the previous expression we have that $\mathrm{Tr}(\bar{\bar{u}}) = i\exp\Big[i(\vec{k}\cdot\vec{r} - \omega t)\Big]\vec{k}\cdot\vec{a} = 0$, so by using Hooke's law, we get:

$$\bar{\bar{\sigma}} = i\exp\Big[i(\vec{k}\cdot\vec{r} - \omega t)\Big]2\mu\left[\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}\begin{pmatrix} a_x & a_y & a_z \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}\begin{pmatrix} k_x & k_y & k_z \end{pmatrix}\right].$$

Due to the fact that $\vec{k} = k\hat{x}$, so $k_y = k_z = 0$, we get:

$$\bar{\bar{\sigma}} = i\exp\Big[i(\vec{k}\cdot\vec{r} - \omega t)\Big]\mu\begin{pmatrix} 0 & a_yk_x & a_zk_x \\ a_yk_x & 0 & 0 \\ a_zk_x & 0 & 0 \end{pmatrix}.$$

Since there are no diagonal terms, there are only shear stresses.

Solution for c):

$$(\boldsymbol{\nabla} \times \vec{u})_i = \varepsilon_{ijk}\partial_j u_k = \varepsilon_{ijk} i k_j u_k = i(\vec{k} \times \vec{u}),$$

but since for longitudinal waves $\vec{k}$ is parallel to $\vec{u}$ (because of their definition), we have that $\boldsymbol{\nabla} \times \vec{u} = 0$.

Solution for d):

$$\vec{u} = a \exp[i(kx - \omega t)]\hat{x}.$$

Therefore:

$$\nabla\vec{u} = aik \exp[i(kx - \omega t)]e_{xx},$$

which means

$$\bar{\bar{u}} = aik \exp[i(kx - \omega t)]e_{xx}.$$

Applying Hooke's law, we have that the only non-zero components of $\bar{\bar{\sigma}}$ are:

$$\sigma_{xx} = (2\mu + \lambda)u_{xx},$$

$$\sigma_{yy} = \sigma_{zz} = \lambda u_{xx}.$$

We have confirmed that $\bar{\bar{\sigma}}$ is diagonal, which means there are only normal stresses (no shearing).