Homework 3
Continuum Mechanics

Adrià Vilanova Martínez

April 11, 2021

Problem 1. One of the hallmarks of the linear regime is the superposition principle. In this problem, we are going to use superposition to find the relation between the bulk modulus K, Young's modulus Y and Poisson's ration ν.

Consider a rectangular block subjected to uniform compressive stresses $-P$ on each face. The equilibrium sides of the block along x, y and z are l, h and w, respectively, and the corresponding length changes are Δl, Δh and Δw.

a) Find the strain along x using Hooke's law as we learnt it in introductory courses (that is, $\sigma_{xx} = Y u_{xx}$ and $u_{yy} = u_{zz} = -\nu u_{xx}$). You will be using superposition since the strain along x is the sum of three contributions, one coming from the compressive stress along x and two additional contributions coming from the compressive stresses along y and z.

   Confirm that by using the inverted version of Hooke's law for isotropic materials we obtain the same answer.

b) The strains along y and z are identical to that along x. Find the bulk modulus of the material as a function of Y and ν. Realize our result is identical to that obtained in class in an alternative way.

Solution for a):
As the only contact force is pressure, the only components of the stress tensor will be the ones in the diagonal, which will be equal to $-P$.

If we only consider the compressive stress along x, we get:

$$u_{xx}^1 = \frac{1}{Y}\sigma_{xx} = -\frac{1}{Y}P.$$

However, we also have to consider the strain caused by the stresses in the y and z directions:

$$\begin{cases} u_{xx}^2 = -\nu u_{yy}^2 = -\dfrac{\nu}{Y}\sigma_{yy} = \dfrac{\nu}{Y}P, \\ u_{xx}^3 = -\nu u_{zz}^3 = -\dfrac{\nu}{Y}\sigma_{zz} = \dfrac{\nu}{Y}P. \end{cases}$$

So, by the superposition principle:

$$u_{xx} = u_{xx}^1 + u_{xx}^2 + u_{xx}^3 = \frac{P}{Y}\left(2\nu - 1\right).$$

We can also calculate the strain along x with the inverted version of Hooke's law for isotropic materials:

$$u_{ij} = \frac{1+\nu}{Y}\sigma_{ij} - \frac{\nu}{Y}\ \mathrm{Tr}\,\bar{\bar{\sigma}}\,\delta_{ij}.$$

By using it, we obtain:

$$u_{xx} = \frac{1}{Y}\left((1+\nu)\sigma_{xx} - \nu\ \mathrm{Tr}\,\bar{\bar{\sigma}}\,\delta_{xx}\right) = \frac{1}{Y}\left(-(1+\nu)P + 3\nu P\right) =$$

$$= \frac{P}{Y}\left(-1-\nu+3\nu\right) = \frac{P}{Y}\left(2\nu-1\right)$$

which is precisely what we got before by using the superposition principle and Hooke's law we learned in introductory courses.

Solution for b):
From the previous result, we obtain:

$$\mathrm{Tr}\,\bar{\bar{u}} = 3P\frac{2\nu-1}{Y} \underset{\hookrightarrow\ \mathrm{Tr}\,\bar{\bar{u}} \approx \frac{\Delta V}{V}}{\Longrightarrow} P\frac{V}{\Delta V} \approx \frac{1}{3}\frac{Y}{2\nu-1}.$$

And by the definition of the bulk modulus, we conclude that

$$K = -V\left(\frac{\partial P}{\partial V}\right)_T \approx -V\frac{\Delta P}{\Delta V} \approx \frac{Y}{3(1-2\nu)}$$

which is the same result we derived in class.

Problem 2. Consider the class example we referred to as "constrained settling". We are going to tackle the problem here in an alternative way to how we did it in class.

a) Start by solving the Navier-Cauchy equation to obtain the displacement field.

b) Then obtain the strain and stress tensors.

c) Why do you think Lautrup refers to this example as an "elastic sea"?

Solution:
The Navier-Cauchy equation is:

$$\vec{f} + \mu\nabla^2\vec{u} + (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) = 0.$$

Since the walls are slippery, they allow vertical but not horizontal displacements. This indicates us that the displacement field will be of the form $\vec{u} = (0, 0, u_z(z))$. By imposing the Navier-Cauchy equation to this expression, we get the following:

$$0 = \begin{pmatrix} 0 \\ 0 \\ -\rho g \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ u_z''(z) \end{pmatrix} + (\lambda+\mu) \begin{pmatrix} 0 \\ 0 \\ u_z''(z) \end{pmatrix} \implies$$

$$\implies \rho g = (\lambda + 2\mu) u_z''(z) \implies u_z''(z) = \frac{\rho g}{\lambda + 2\mu} =: \frac{1}{D} \implies$$

$$\implies u_z'(z) = \frac{1}{D} z + C \implies u_z(z) = \frac{1}{2D} z^2 + Cz + B$$

where C, B are constants that are determined by the boundary conditions.

In our case, our boundary conditions are the following ones:

$$\begin{cases} u_z(0) = 0 \implies B = 0 \\ [\bar{\bar{\sigma}} \cdot \hat{n}] = 0. \end{cases}$$

Let's calculate the strain tensor:

$$\bar{\bar{u}} = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$$

where

$$\nabla\vec{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{z}{D} + C \end{pmatrix}.$$

Therefore:

$$\bar{\bar{u}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{z}{D} + C \end{pmatrix}.$$

We can then calculate the stress vector using Hooke's law:

$$\sigma_{ij} = 2\mu u_{ij} + \lambda \operatorname{Tr} \bar{\bar{u}}\, \delta_{ij} \implies$$

$$\implies \bar{\bar{\sigma}} = \left(\frac{z}{D} + C\right)\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda + 2\mu \end{pmatrix}$$

Now we can impose the boundary condition $[\bar{\bar{\sigma}} \cdot \hat{n}] = 0$. Since the interface is a free surface (we neglect atmospheric pressure), we have that $\bar{\bar{\sigma}}_2 = 0$, which means that:

$$\bar{\bar{\sigma}}_1 \hat{n} = 0 \implies \sigma_{iz}(h) = 0 \quad \forall i.$$

In particular, $\sigma_{zz}(h) = 0$, and this means

$$C = -\frac{h}{D}.$$

In conclusion, substituting C and D into the expressions we found:

$$\begin{cases} \vec{u} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{D}\left(\frac{1}{2}z^2 - hz\right) \end{pmatrix} \\ \bar{\bar{u}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{D}(z-h) \end{pmatrix} \\ \bar{\bar{\sigma}} = \frac{\lambda}{D}(z-h)\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda + 2\mu \end{pmatrix} \end{cases}.$$

Lautrup refers to this example as an elastic sea because $p_z = -\sigma_{zz} = \rho g(h-z)$, and p_z increases linearly with depth like a fluid at rest.

Problem 3. Consider the situation studied in problem 2: a homogeneous and isotropic elastic solid with denisty ρ inside a container with rigid walls. However, in this case, the container is cylindrical and not a prism, and there is friction between the elastic solid and the walls. Hence, $\sigma_{zr}(r=R) = \mu_s \sigma_{zz}(r=R)$, where R is the radius of the circular cross-section and μ_s is the friction coefficient. There is still gravity; the corresponding specific force is $\vec{g} = g\hat{z}$. Note the z-axis runs along the cylinder and points downwards.

a) Discuss how friction affects both the strain and stress tensors. What new components arise in these tensors relative to the case without friction?

b) Now consider a thin slice of elastic material inside the cylindrical container; this slice will have cross-sectional area πR^2 and height Δz.

   Apply the condition of mechanical equilibrium to this slice, assuming that σ_{zz} is uniform across the circular cross-section and that $\sigma_{rr} = \sigma_{\theta\theta} = k\sigma_{zz}$, with k a constant, and show that

$$\frac{d\sigma_{zz}}{dz} = -\rho g - \frac{2\mu_s k}{R}\sigma_{zz}.$$

c) Integrate this equation and show that

$$p_z = \rho g \lambda \left(1 - e^{-\frac{z}{\lambda}}\right)$$

   where $\lambda = \frac{R}{2\mu_s k}$. Note we have assumed that at the top of the solid $\sigma_{zz} = 0$.

d) Consider the limiting cases $z \ll \lambda$ and $z \gg \lambda$ and discuss the physics in each case. Doing this illustrates the physical significance of λ and what friction brings to the problem.

Solution for a):
Due to friction, $\vec{u} = (0, 0, u_z(r,z))$, so in adition to the dependence on z, there's now an r-dependence too.

$$\vec{\nabla}\vec{u} = \begin{pmatrix} 0 & 0 & \partial_r u_z \\ 0 & 0 & 0 \\ 0 & 0 & \partial_z u_z \end{pmatrix} \implies \bar{\bar{u}} = \frac{1}{2}(\vec{\nabla}\vec{u} + (\vec{\nabla}\vec{u})^T) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\partial_r u_z \\ 0 & 0 & 0 \\ \frac{1}{2}\partial_r u_z & 0 & \partial_z u_z \end{pmatrix}.$$

We can now use Hooke's law:

$$\bar{\bar{\sigma}} = 2\mu \bar{\bar{u}} + \lambda(\mathrm{Tr}\,\bar{\bar{u}})\bar{\bar{I}} \implies$$

$$\bar{\bar{\sigma}} = \begin{pmatrix} \lambda\partial_z u_z & 0 & \mu\partial_r u_z \\ 0 & \lambda\partial_z u_z & 0 \\ \mu\partial_r u_z & 0 & (2\mu+\lambda)\partial_z u_z \end{pmatrix}.$$

Solution for b):
The local condition for mechanical equilibrium is

$$\vec{f} + \nabla \cdot \bar{\bar{\sigma}}^T = 0 \quad \forall \text{ material particles.}$$

If we integrate over volume V:

$$\int_V \vec{f} dv' + \int_V \vec{\nabla} \cdot \bar{\bar{\sigma}}^T dv' = \int_V \vec{F} dv' + \oint_A \bar{\bar{\sigma}} \cdot d\vec{S} = 0,$$

where V is a cylindrical slice of the material ($|V| = \pi R^2 \Delta z$).

$$\int_V \vec{f} dv' = \int_V \rho \vec{g} dv' = \hat{e}_z \rho g \pi R^2 \Delta z.$$

$$\oint_A \bar{\bar{\sigma}} \cdot d\vec{S} = \int_{A_1} \bar{\bar{\sigma}} d\vec{S} + \int_{A_2} \bar{\bar{\sigma}} d\vec{S} + \int_{A_\perp} \bar{\bar{\sigma}} d\vec{S},$$

where A_1, A_2 are the circular surfaces at the top and at the bottom, and $A_\perp$ is the lateral surface.

For the top and bottom surfaces, we have:

$$\int_{A_i} \bar{\bar{\sigma}} \cdot d\vec{S} = \int_{A_i} (-1)^i \bar{\bar{\sigma}}_{rz}(z_i) \hat{e}_r r d\theta dr + \int_{A_i} (-1)^i \sigma_{zz}(z_i) \hat{e}_z r d\theta dr,$$

and since $\sigma_{zz} \neq f(r)$ because we've supposed that it is uniform across the entire section, we have:

$$\int_{A_i} \bar{\bar{\sigma}} \cdot d\vec{S} = 0 + \int_{A_i} (-1)^i \sigma_{zz}(z_i) \hat{e}_z r d\theta dr = (-1)^i \hat{e}_z \sigma_{zz}(z_i) R^2 \pi$$

For the lateral surface, we have:

$$\int_{A_\perp} \bar{\bar{\sigma}} \cdot d\vec{S} = \int_{A_\perp} \sigma_{rr} R d\theta dz \hat{e}_r + \int_{A_\perp} \sigma_{zr}(R) R d\theta dz \hat{e}_z =$$

$$= 0 + \mu_S K R \hat{e}_z \int_z^{z+\Delta z} \sigma_{zz} dz' \int_0^{2\pi} d\theta \approx 2\pi R \mu_S k \hat{e}_z \sigma_{zz} \Delta z.$$

Putting it all together:

$$\hat{e}_z(\rho g \pi R^2 \Delta z - \sigma_{zz}(z) \pi R^2 + \sigma_z zz + \Delta z \pi R^2 + 2\pi R \mu_S k \sigma_{zz}\ \Delta z) = 0 \implies$$

$$\implies \pi R^2 \frac{\sigma_{zz}(z + \Delta z) - \sigma_{zz}(z)}{\Delta z} = -\rho g \pi R^2 - 2\pi R \mu_s k \sigma_{zz}.$$

Taking the limit for $\Delta z \to 0$:

$$\pi R^2 \frac{d\sigma_{zz}}{dz} = -\rho g \pi R^2 - 2\pi R \mu_S k \sigma_{zz} \implies$$

$$\implies \frac{d\sigma_{zz}}{dz} = -\rho g - \frac{2\mu_s k}{R} \sigma_{zz}.$$

Solution for c):
We'll first solve the homogeneous ODE

$$f'(z) + \frac{1}{\lambda} f(z) = 0.$$

This is a linear ODE with known solution

$$f_h(z) = e^{-\frac{z}{\lambda}}.$$

We can then solve the full ODE with the constant variations method: let's suppose the solution to the full ODE is $\sigma_{zz}(z) = f(z) = f_h(z)g(z)$. Then:

$$f_h'(z)g(z) + f_h(z)g'(z) = f'(z) = -\rho g - \frac{1}{\lambda} f_h(z)g(z) \implies$$

$$\implies \cancel{-\frac{1}{\lambda} f_h(z)g(z)} + g'(z)f_h(z) = -\rho g \cancel{- \frac{1}{\lambda} f_h(z)g(z)} \implies$$

$$\implies g'(z) = -\rho g e^{\frac{z}{\lambda}} \implies g(z) = C - \rho g \lambda e^{\frac{z}{h}} = C - \rho g \lambda \frac{1}{f_h(z)} \implies$$

$$\implies \sigma_{zz}(z) = f(z) = f_h(z)\left(C - \rho g \lambda \frac{1}{f_h(z)}\right) = C f_h(z) - \rho g \lambda = C e^{-\frac{z}{h}} - \rho g \lambda.$$

If we impose $\sigma_{zz}(0) = 0$:

$$0 = \sigma_{zz}(0) = C - \rho g \lambda \implies C = \rho g \lambda,$$

and thus:

$$\sigma_{zz}(z) = \rho g \lambda \left(e^{-\frac{z}{h}} - 1\right).$$

Since $p_z = -\sigma_{zz}$, we have shown that the statement is true.

Solution for d):
$\lambda = \frac{R}{2\mu_S k}$ is a characteristic length scale (μ_S, k are dimensionless). Using $\mu_s \approx 0.5$, $k \approx 1$, we find $\lambda \approx R$.

To discuss the case in which $z \ll \lambda$, we can approximate p_z via a truncated Taylor series:

$$p_z(z) = \rho g \lambda \left(\frac{z}{\lambda} + \mathcal{O}\left(\frac{z}{\lambda}\right)^2\right) \approx \rho g z,$$

which is the behavior in the absence of friction.

If instead $z \gg \lambda$, this means $\frac{z}{\lambda} \gg 1$ and therefore $e^{-\frac{z}{h}} \ll 1$. Thus:

$$e^{-\frac{z}{h}} \approx 0 \implies p_z(z) \approx \rho g \lambda,$$

which is a constant value.

Therefore, we can conclude the pressure saturates for sufficiently large z. Friction supports the weight of the elastic material for sufficiently light columns.